MULTI PRIZES FOR MULTI TASKS: EXTERNALITIES AND THE OPTIMAL DESIGN OF TOURNAMENTS

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October 11, 2018

Abstract: This paper studies a multi-task tournament in which each agent undertakes two tasks. An agent's effort on one task creates externalities on the performance of the other task of the agent as well as the performances of other competing agents. We discuss the design of an optimal tournament to achieve a social optimum in the presence of such externalities. In particular, we show that the traditional single-prized tournament is unable to elicit a social optimum, while a task-specific, multi-prized tournament, which we propose in this paper, can achieve socially optimal outcomes.

Keywords: Tournament; Externality; Incentive Mechanism; Multi Task; Single-prized Tournament; Multi-prized Tournament

JEL Classification Numbers: D60, J33, M50

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In the principal-agent problem, the principal needs to use certain mechanisms to induce the optimal effort levels from the agents. The design of incentive schemes is an important issue to explore. Tournament is such a scheme(Lazear and Rosen, 1981). A tournament involves several agents with each undertaking one task to produce a single product. It awards the participating agents on the basis of the ordinal ranking of their performances. In equilibrium, the effort levels that agent make are optimal. The optimum property with agents performing one task and producing one output can also be preserved when the model is extended to the setting in which agents undertake multiple tasks and produce multiple outputs (see, for example, Holmstrom and Milgrom, 1991; Barlevy and Neal, 2012; Liu and Xu, 2018).

In the tournament settings discussed above, there are no *inter-agent externalities* in which an agent's actions may affect the performances of other competing agents, nor are there *inter-task externalities* in which the performance of a task of one agent may be affected by this agent's actions on other tasks. It has been noted that, in many situations, there are various externalities in the performances of competing agents. These externalities can take various forms. For example, agents may engage in sabotage activities to increase the probability of winning by reducing their opponents' measured output (Lazear, 1989), or an agent may help co-workers 'as with on-the-job training of junior by senior employees' (Drago and Garvey, 1998), or due to team externalities, greater efforts by one agent increase another agent's output (Drago and Turnbull, 1988). In multi-task environments, in some cases, an agent's effort on one task may affect the performance of this agent as well as other agents on other tasks. One example is the research management problem (Bardsley,1999). Agents (scientist) allocate effort and resources among multiple tasks (a portfolio of projects), and the principal (the central research manager) uses the allocation of funds as the incentive to achieve policy objectives. Scientists engage in a tournament to get funds. Some projects may have external effects on other projects due to the spillover of scientific knowledge. Political competition and election is another example. Voters, as the principal, elect the 'best' candidate for the office or position. Political candidates are agents and perform multiple tasks to get elected. The externalities across tasks and across candidates may exist, e.g., the introduction of a refinery factory promotes local economic development, but may damage the local environment and the environment of the neighboring districts.

In the literature on tournaments with a single task and a single output, it has been shown that, in the presence of inter-agent externalities, tournaments often fail to achieve their intended goals (see Lazear, 1989, Drago and Turnbull, 1988, for some early contributions, and Chowdhury and Guertler, 2015, and Connelly, Tihanyi, Crook, and Gangloff, 2014 for surveys on the related contributions). On the other hand, there seems no study in the literature that explores tournaments with multi tasks and multi outputs in the presence of inter-task and/or inter-agent externalities. This paper tries to fill this gap and studies tournaments in which there are inter-agent as well as inter-task externalities.

might be confusing of using the two terms

In our set-up, each agent has two tasks to undertake, and one of the tasks produce both inter-agent and inter-task externalities. It has effect on the performances of the other task of the self agent and the competing agents as well. We examine the problem of designing tournaments to induce the optimal effort levels from the agents, and show that, in the presence of inter-agent and inter-task externalities, there is no single-prized tournament that can be used to elicit the optimal effort levels. We thus propose the task-specific, multi-prized tournaments, which can accomplish the intended goal in this case. In our design, for each task, agents are ranked according to their performances along this task, and task-specific prizes are awarded to the agents based on their performances on each task. Each agent receive multiple prizes from multiple tasks. Through adjusting the spread of the prizes for different tasks, optimal effort levels from the agent can be induced. The intuition that task-specific multi-prized tournaments can induce optimal effort levels may be explained as follows. In a tournament with multiple tasks, competing agents exert effort to increase the rank of their performances and balance the efforts among different tasks. If a task has spillover effects on the performance of other competing agents, the incentive of tournaments is distorted due to that agents do not internalize such externalities. To increase performance rank, an agent tends to put too much (or too little) efforts on tasks that have negative (or positive) externalities on other agents to reduce their measured output. The task-specific, multi-prized tournaments can resolve this distortion by adjusting the prizes for different tasks. A low (high) winning prize for a task with negative externalities reduces (increases) the agents' efforts and adjust efforts levels to the optimum.

The remainder of the paper is organized as follows. In Section 1, we introduce and set up our model. Section 2 discusses the design of the tournaments and presents our main results. Section 3 contains a few concluding remarks.

1. The Model

1.1. **The setup.** In this section, we present our basic model. Consider two competing agents, 1 and 2, in a tournament. The agents choose their efforts for two tasks: a task with externality, to be called e, and a task without externality, to be called t. For each agent $i \in \{1, 2\}$, let e_i and t_i be the effort levels i spends on e and on t, respectively. Agent i's ($i \in \{1, 2\}$) 'production functions' are assumed to take the following forms:

$$E_i = e_i + \epsilon_i \tag{1.1}$$

$$T_i = \alpha t_i + \beta e_i + \beta' e_j + \xi_i \tag{1.2}$$

where E_i and T_i, respectively, measure agent i's performance on e and t. $\alpha (> 0)$, β and β' are given parameters. ϵ_1, ϵ_2, ξ_1 and ξ_2 are random variables with zero means and are assumed to be independently and identically distributed (i.i.d.).

It may be remarked that, in the above production functions, the parameter β captures the *inter-task externality*. The cross-task externality can be interpreted as an externality incurred by an agent's effort in task e on the performance of the same agent's task t. β' captures the *inter-agent externality* and represents externalities imposed by an agent's effort in task e on the other agent's performance of task t.

For each agent $i \in \{1, 2\}$, let $C(e_i, t_i)$ be the cost function when i exerts effort levels e_i and t_i. We assume that the cost function is the same for both agents. The cost function $C(\cdot, \cdot)$ is assumed to be strictly increasing in each of its arguments, strictly convex, $C(0, 0) = 0$, $C_{e_i}(0, a) = C_{t_i}(b, 0) = 0$ for all

$a \geq 0, b \geq 0$, and $\lim_{a\to\infty} C_{e_i}(a, t_i) \to \infty$ for all $t_i > 0$ and $\lim_{b\to\infty} C_{t_i}(e_i, b) \to \infty$ for all $e_i > 0$.

1.2. **Optimal choices of effort levels.** We first consider socially optimal choices of effort levels by the agents. For this purpose, we consider the principal's problem where the principal chooses agents' effort levels to maximize the expected value of a simple sum of outputs net the costs of exerting such efforts:

$$\max_{e_1,e_2,t_1,t_2} E[E_1 + T_1 + E_2 + T_2 - C(e_1, t_1) - C(e_2, t_2)] \tag{1.3}$$

Substituting the outputs and taking the expectation, we have

$$\max_{e_1,e_2,t_1,t_2} (1 + \beta + \beta')(e_1 + e_2) + \alpha(t_1 + t_2) - C(e_1, t_1) - C(e_2, t_2) \tag{1.4}$$

Let e_1^*, e_2^*, t_1^* and t_2^* be the solutions to the above problem. Then, noting that the objective function of problem (1.4) is strictly concave in e_i and t_i, the following first order conditions are both necessary and sufficient for $i = 1, 2$:

$$\begin{aligned} (1 + \beta + \beta') - C_{e_i}(e_i^*, t_i^*) &\leq 0 \quad (= 0 \text{ if } e_i^* > 0) \\ \alpha - C_{t_i}(e_i^*, t_i^*) &\leq 0 \quad (= 0 \text{ if } t_i^* > 0) \end{aligned}$$

It may be noted that, if $(1 + \beta + \beta') \leq 0$, then $e_1^* = 0 = e_2^*$. That is, the optimal choices of effort levels for task e are 0 for the agents. The intuition is fairly straightforward: when externalities inflicted on task t in performing task e by the agents are destructive and large (so that $\beta+\beta' \leq -1$), it is optimal for the principal to ask the agents to perform just one task, task t, which causes no externalities. In this case, the problem is reduced to the conventional single task problem. In the subsequent discussions, therefore, we consider the case in which $1 + \beta + \beta' > 0$.

Proposition 1. *Let $1 + \beta + \beta' > 0$. Then, there exists a unique set of effort levels, $(e_1^*, e_2^*, t_1^*, t_2^*)$, to the problem (1.4) such that $e_1^* > 0, e_2^* > 0, t_1^* > 0$ and $t_2^* > 0$.*

Proof. Let $1 + \beta + \beta' > 0$.

For each $i = 1, 2$, define a function $h(e_i, t_i) = (1 + \beta + \beta')e_i + \alpha t_i - C(e_i, t_i)$. Since $C(e_i, t_i)$ is strictly convex, $h(e_i, t_i)$ is strictly concave. Note that $C_{e_i}(0, t_i) = 0$ for all $t_i \geq 0$, $C_{t_i}(e_i, 0) = 0$ for all $e_i \geq 0$, $\lim_{a \to \infty} C_{e_i}(a, t_i) \to \infty$ for all $t_i > 0$ and $\lim_{b \to \infty} C_{e_i}(e_i, b) \to \infty$ for all $e_i > 0$.

Let $U = \{\mathbf{u} \in \mathbb{R}^2_+ : u_1 + u_2 = 1\}$. Any vector (e_i, t_i) can be uniquely expressed as $\lambda\mathbf{u}$ for some $\lambda \geq 0$ and some $\mathbf{u} \in U$. Given our assumptions on $C(e_i, t_i)$, for any given vector $\mathbf{u} \in U$, it must be the case that $C(\lambda\mathbf{u})$ is increasing and convex in λ, and $\lim_{\lambda \to \infty} C_\lambda(\lambda\mathbf{u}) = \infty$. Since $(1 + \beta + \beta')\lambda e_i + \alpha\lambda t_i$ is concave in λ, for any $\mathbf{u} \in U$, there exists a finite cutoff value, λ_u, of λ such that $h(e_i, t_i)$ evaluated at $(e_i, t_i) = \lambda\mathbf{u}$ will be negative for all $\lambda \geq \lambda_u$. Let $\lambda^* = sup\{\lambda_u : \mathbf{u} \in U\}$. Since U is compact, λ^* is well defined and finite. It follows that the global maximum for $h(e_i, t_i)$ lies in the bounded set $[0, \lambda^*]^2$.

Since the function $h(e_i, t_i)$ is strictly concave, it is also strictly concave over $[0, \lambda^*]^2$, which is the region that contains the global optimum. This ensures that the following first-order conditions are both necessary and sufficient a global maximum:

$$(1 + \beta + \beta') - C_{e_i}(e_i^*, t_i^*) \leq 0 (= 0 \text{ if } e_i^* > 0)$$
$$\alpha - C_{t_i}(e_i^*, t_i^*) \leq 0 (= 0 \text{ if } t_i^* > 0)$$

The boundary conditions on C and subsequently on h, together with the assumptions that $(1 + \beta + \beta') > 0$ and $\alpha > 0$, ensure the maximum of h being achieved at an interior point so that $e_i^* > 0$ and $t_i^* > 0$. Therefore, there exist $e_i^* > 0, t_i^* > 0$ $(i = 1, 2)$ satisfying the following equations:

$$(1 + \beta + \beta') - C_{e_i}(e_i^*, t_i^*) = 0$$
$$\alpha - C_{t_i}(e_i^*, t_i^*) = 0$$

These are necessary and sufficient conditions for the problem (1.4). Since h is strictly concave, $e_i^*, t_i^* (i = 1, 2)$ are unique. □

Subsequently, we shall call $(e_1^*, t_1^*, e_2^*, t_2^*)$ that solves the problem (1.4) as the 'social optimum'. Since the principal does not observe the agents' choices of effort levels, we shall introduce incentive schemes needed to induce the social optimum in the next section.

2. Tournaments

In this section, we discuss the design of tournaments to achieve social optimum characterized in the last section. Two different forms of tournament will be explored: a single-prized tournament and a multi-prized tournament. In a single-prized tournament, there is one tournament for both tasks combined and a single prize will be given to the winner, while in a multi-prized tournament, there is a tournament for each task and a prize will be given to the winner of each tournament.

2.1. **Single-prized tournament.** In this subsection, we discuss single-prized tournaments. A single-prized tournament involves a *bonus*, to be denoted by B, and a base pay, to be denoted by B_0. In our discussion, we do not restrict B to be positive only. The bonus B is given to the agent who has a bigger total output than the other agent where the total output is the simple sum of the agent's performances on the two tasks. We first discuss the design of B by the principal.

We model the two agents as playing a simultaneous move game. Agent i's ($i \in \{1,2\}$) objective is to solve the following maximization problem given the other agent's choices:

$$\max_{e_i,t_i} \quad B_0 + BPr[E_i + T_i > E_j + T_j] - C(e_i,t_i) \tag{2.1}$$

Note that

$$E_i + T_i > E_j + T_j \Leftrightarrow \epsilon_j + \xi_j - \epsilon_i - \xi_i < (1+\beta-\beta')(e_i - e_j) + \alpha(t_i - t_j)$$

Then, the above optimization problem (2.1) can be rewritten as follows:

$$\max_{e_i,t_i} \quad B_0 + BPr[\epsilon_j + \xi_j - \epsilon_i - \xi_i < (1+\beta-\beta')(e_i - e_j) + \alpha(t_i - t_j)] - C(e_i,t_i) \tag{2.2}$$

Let $G(\cdot)$ be the cumulative distribution function (cdf) of the random variable $\epsilon_i + \xi_i - \epsilon_j - \xi_j$. Then the optimization problem for agent i ($i \in \{1,2\}$ is:

$$\max_{e_i,t_i} \quad B_0 + BG[(1+\beta-\beta')(e_i - e_j) + \alpha(t_i - t_j)] - C(e_i,t_i) \tag{2.3}$$

Let $G(\cdot)$ be differentiable with $G'(\cdot) = g(\cdot)$. Agent i's best responses to the competing agent's efforts can be characterized by the following first order conditions:

$$(1+\beta+\beta')Bg[(1+\beta-\beta')(e_i-e_j)+\alpha(t_i-t_j)]-C_{e_i}(e_i,t_i) \leq 0(=0 \text{ if } e_i>0) \quad (2.4)$$

$$\alpha Bg[(1+\beta-\beta')(e_i-e_j)+\alpha(t_i-t_j)]-C_{t_i}(e_i,t_i) \leq 0(=0 \text{ if } t_i>0) \quad (2.5)$$

Let $((e_1^s, t_1^s), (e_2^s, t_2^s))$ denote a Nash equilibrium pair of efforts chosen by the two agents.

Proposition 2. *For each B, there exists a symmetric Nash equilibrium pair of efforts which involves both agents choosing the same effort levels: $e_1^s = e_2^s \geq 0, t_1^s = t_2^s \geq 0$.*

Proof. For a given B, a Nash equilibrium, $((e_1^s, t_1^s), (e_2^s, e_2^s))$, is a solution that solves the agents' best responses, (2.4), (2.5). Being symmetric, the solution is such that $e_1^s = e_2^s, t_1^s = t_2^s$ and satisfies

$$(1+\beta-\beta')Bg(0)-C_{e_1}(e_1,t_1) \leq 0(=0 \text{ if } e_1>0) \quad (2.6)$$

$$\alpha Bg(0)-C_{t_1}(e_1,t_1) \leq 0(=0 \text{ if } t_1>0) \quad (2.7)$$

When $B=0$, from the above, $e_1^s = e_2^s = 0$ and $t_1^s = t_2^s = 0$ solve the problem.

When $B<0$, from equation (2.7), we have $t_1^s = 0$. If $(1+\beta-\beta') \geq 0$, then $(e_1^s = 0, t_1^s = 0)$ satisfies (2.6) and (2.7). When $1+\beta-\beta' < 0$, (2.6) becomes

$$(1+\beta+\beta')Bg(0)-C_{e_1}(e_1,0)=0$$

Note that $(1+\beta+\beta')Bg(0) > 0$. Given the boundary conditions of $C(e_1,t_2)$, there is $e_1^s > 0$ satisfying the above condition. Hence, in this case, there exists a pair $(e_1^s > 0, t_1^s = 0)$ satisfying (2.6) and (2.7).

Consider $B>0$. Suppose first $1+\beta-\beta' > 0$. Then, following a similar proof strategy to that of Proposition 1, we can show that there exist $e_1^s > 0$, $t_1^s > 0$ satisfying (2.6) and (2.7). If $1+\beta-\beta' \leq 0$, then, from (2.6), $e_1^s = 0$. Given the conditions on $C(e_1,t_1)$, from (2.7), there exists $t_1^s \geq 0$ that satisfies (2.7). □

Proposition 2 informs us the existence of a symmetric Nash equilibrium. As we have seen in the process of proving Proposition 2, the question whether a single-prized tournament will be able to elicit optimal efforts from the agents lingers, and the answer to this question may depend on the parameters β and β'. In the rest of this subsection, we discuss whether a single-prized tournament can accomplish its intended goal of eliciting optimal efforts from the agents.

Proposition 3. *Let $1+\beta+\beta' > 0$. If $\beta' = 0$, then there exists a $B > 0$ such that the symmetric Nash equilibrium of the single-prized tournament is the social optimum, e.g., $(e_i^s, t_i^s) = (e_i^*, t_i^*)$ for $i = 1, 2$.*

Proof. When $\beta' = 0$, $1+\beta+\beta' = 1+\beta > 0$. The first order conditions (2.6) and (2.7) for an interior symmetric Nash equilibrium become: for each $i = 1, 2$,

$$\begin{aligned}(1+\beta)Bg[0] - C_{e_i}(e_i^s, t_i^s) &= 0\\ \alpha Bg[0] - C_{t_i}(e_i^s, t_i^s) &= 0\end{aligned}$$

On the other hand, the social optimum, $(e_i^*, t_i^*)(i = 1, 2)$, is characterized by the following:

$$\begin{aligned}(1+\beta) - C_{e_i}(e_i^*, t_i^*) &= 0\\ \alpha - C_{t_i}(e_i^*, t_i^*) &= 0\end{aligned}$$

By setting $B = 1/g[0]$ and from the proof of Proposition 1 that the social optimum is unique, we must have $(e_i^s, t_i^s) = (e_i^*, t_i^*)$ for $i = 1, 2$. □

Proposition 3 stats that, if there is no cross-agent externality, a single-prized tournament can achieve the social optimum. The optimum is obtained by the agents' internalization of the cross-task externalies. However, when there are cross-agent externalities, a single-prized tournament fails to induce social optimal efforts, as shown by the following proposition.

Proposition 4. *Let $1+\beta+\beta' > 0$. If $\beta' \neq 0$, then there exists no B such that the symmetric Nash equilibrium of the single-prized tournament is the social optimum.*

Proof. Let $(1+\beta+\beta')>0$ and $\beta'\neq 0$. We note that if there was a B such that $(e_i^s,t_i^s)=(e_i^*,t_i^*)$ for $i=1,2$. Then, we would have

$$(1+\beta-\beta')Bg[0]-C_{e_i}(e_i^s,t_i^s)=0$$
$$\alpha Bg[0]-C_{t_i}(e_i^s,t_i^s)=0$$

and

$$\begin{aligned}(1+\beta+\beta')-C_{e_i}(e_i^*,t_i^*) &= 0\\ \alpha-C_{t_i}(e_i^*,t_i^*) &= 0\end{aligned}$$

From the above, we would then obtain

$$(1+\beta+\beta')=C_{e_i}(e_i^*,t_i^*)=C_{e_i}(e_i^s,t_i^s)=(1+\beta-\beta')Bg[0] \tag{2.8}$$
$$\alpha=C_{t_i}(e_i^*,t_i^*)=C_{t_i}(e_i^s,t_i^s)=\alpha Bg[0] \tag{2.9}$$

(2.8) would imply

$$1+\beta+\beta'=(1+\beta-\beta')Bg[0]. \tag{2.10}$$

and (2.9) would imply

$$1=Bg[0] \tag{2.11}$$

(2.10) and (2.11) would be in contradiction with $\beta'\neq 0$. Therefore, there is no B such that $(e_i^s,t_i^s)=(e_i^*,t_i^*)$ for $i=1,2$ □

When there are externalities across competing agents, i.e., when $\beta'\neq 0$, a single-prized tournament cannot achieve the social optimum. To understand the intuition behind this result, we note that, in a single-prized tournament, the winning agent is the one who produces the greatest 'total output', the total output being the simple sum of the performances of the two tasks. In the production functions of the agent, the task t has no externalities while the task e creates externalities on the agent's performance in task t and the competing agent's task t as well. The social optimum is obtained by internalizing these externalities. However, when the agents are engaged in a single tournament, though the agent can internalize externalities across tasks, the externalities across the agents are ignored in calculating Nash equilibrium choices of efforts. As a consequence, the externalities across the agents cannot be internalized, and consequently, a single-prized tournament cannot achieve the social optimum.

2.2. **Multi-prized tournament.** As shown in Section 3.1, there is a difficulty in using a single-prized tournament to achieve the social optimum in the presence of cross-agent externalities (i.e., when $\beta' \neq 0$). In this Section, we introduce and consider an alternative tournament scheme, a multi-prized tournament, and examine whether it can be used by the principal to achieve the social optimum.

A multi-prized tournament consists of two separate 'tournaments', to be called an e-tournament and a t-tournament, for the two agents to compete for. An e-tournament is for the performance of task e and a t-tournament is designed for the performance of task t. The winner of each tournament is determined by the relative performance of each task. Let B_e and B_t, respectively, be the prizes for the e-tournament and t-tournament. Again, let B_0 be the base payment to the agent.[1]

The two agents play a simultaneous-move game in which they each choose a pair of efforts (e_i, t_i) $(i = 1, 2)$ to maximize the expected payoffs. Specifically, each agent i $(i = 1, 2)$ solves the following problem:

$$\max_{e_i, t_i} \quad B_0 + B_e Pr[E_i > E_j] + B_t Pr[T_i > T_j] - C(e_i, t_i) \tag{2.12}$$

Let $((e_1^m, t_1^m), (e_2^m, t_2^m))$ denote a Nash equilibrium pair of choices of efforts by the two agents when they play the game in this Section. Then, we obtain the following results summarized in Propositions 5 and 6.

Proposition 5. *For suitably chosen B_e and B_t, there exists a unique symmetric Nash equilibrium pair of choices of efforts which involves both agent choosing the same effort levels $e_1^m = e_2^m > 0, t_1^m = t_2^m > 0$.*

Proof. Note that

$$E_i > E_j \Leftrightarrow \epsilon_j - \epsilon_i < e_i - e_j$$

[1]In this paper, we focus on the design of B_e and B_t. The choice of B_0 is done by considering the participation constraints of the agents once B_e and B_t are determined.

and

$$T_i > T_j \Leftrightarrow \xi_j - \xi_i < \alpha(t_i - t_j) + (\beta - \beta')(e_i - e_j)$$

So, the above problem (2.12) can be rewritten as follows:

$$\max_{e_i,t_i} \quad B_e Pr[\epsilon_j - \epsilon_i - < e_i - e_j] + B_t Pr[\xi_j - \xi_i < \alpha(t_i - t_j) + (\beta - \beta')(e_i - e_j)] - C(e_i, t_i)$$

Let $H_E(\cdot)$ be the cdf of the random variable $\epsilon_j - \epsilon_i$ and $H_T(\cdot)$ be the cdf of the random variable $\xi_j - \xi_i$. Then, the above can be rewritten as the following:

$$\max_{e_i,t_i} \quad B_e H_E[e_i - e_j] + B_t H_T[\alpha(t_i - t_j) + (\beta - \beta')(e_i - e_j)] - C(e_i, t_i)$$

Let $H'_E(\cdot) = h_E(\cdot)$ and $H'_T(\cdot) = h_T(\cdot)$. Considering symmetric equilibrium choices of effort levels, we obtain the following

$$B_e h_E[0] + (\beta - \beta')B_t h_T[0] - \frac{\partial C(e_i, t_i)}{\partial e_i} \quad \leq \quad 0 (= 0 \text{ if } e_i > 0) \tag{2.13}$$

$$\alpha B_t h_T[0] - \frac{\partial C(e_i, t_i)}{\partial t_i} \quad \leq \quad 0 (= 0 \text{ if } t_i > 0) \tag{2.14}$$

Following a similar proof strategy to that of Proposition 2, it can be shown that, if B_e and B_T are chosen such that $B_e h_E[0] + (\beta - \beta')B_t h_T[0] > 0$, then there are $e_1^m = e_2^m > 0, t_1^m = t_2^m > 0$ such that

$$B_e h_E[0] + (\beta - \beta')B_t h_T[0] = \frac{\partial C(e_i^m, t_i^m)}{\partial e_i} \tag{2.15}$$

$$\alpha B_t h_T[0] = \frac{\partial C(e_i^m, t_i^m)}{\partial t_i} \tag{2.16}$$

It may be noted (as in the proof of Proposition 2) that the solution to the system of equations, (2.15) and (2.16), is unique. □

Proposition 6. *There exist B_e and B_t such that $(e_i^m, t_i^m) = (e_i^*, e_i^*)$ for $i = 1, 2$.*

Proof. From proposition 5, there are $e_1^m = e_2^m > 0, t_1^m = t_2^m > 0$ satisfying (2.15) and (2.16). On the other hand, we have

$$(1+\beta+\beta') = C_{e_i}(e_i^*, t_i^*)$$
$$\alpha = C_{t_i}(e_i^*, t_i^*)$$

If we set $B_e h_E[0] = 1+2\beta'$ and $B_t h_T[0] = 1$, then, $B_e h_E[0] + (\beta-\beta')B_t h_T[0] = 1+2\beta' + (\beta-\beta') = 1+\beta+\beta' > 0$, and consequently, $(e_i^m, t_i^m) = (e_i^*, e_i^*)$ for $i = 1, 2$. □

Therefore, a multi-prized tournament can be used by the principal to induce the optimal effort levels from the agents. From the proof of Proposition 2.17, the choices of task-specific 'prizes' are:

$$B_t = \frac{1}{h_T(0)} \quad \text{for the T task,}$$

discuss more on the equation. How h() affect prize

$$B_e = \frac{1+2\beta'}{h_E(0)} \quad \text{for the E task}$$

It may be noted that $B_t > 0$, while B_e can be positive, or negative, or zero depending on the parameter β' of the cross-agent externalities:

$$\beta' \geq -1/2 \text{ if and only if } B_e \geq 0.$$

Note that the sign and size of B_e depend on β', the parameter capturing the cross-agent externalities. This can be intuitively understood as a way that the principal internalizes such externalities. In particular, if such externalities are negative and significant, then, in the design of B_e, the principal uses a task-specific negative prize to curb such detrimental activities to achieve optimality. The flexibility of choosing both prizes, B_t and B_e, enables the principal to internalize cross-agent externalities. This is in sharp contrast to a single-prized tournament where the principal does not have this kind of flexibility, and, as a consequence, when $\beta' \neq 0$, a single-prized tournament cannot induce the optimal effort levels from the agents.

3. Conclusion

In this paper, we consider the problem of designing tournaments to induce the optimal effort levels from competing agents. Agents perform multiple tasks and produce many outputs, and there are inter-agent and inter-task externalities. We have shown that, in such environments, a single-prized tournament fails to induce the optimal effort levels from the agents, while task-specific multi-prized tournaments can be used to induce the agents to choose the optimal levels of effort.

An implication of our analysis and results is that when agents perform multiple tasks and produce multiple outputs, and when there are inter-agent and inter-task externalities, the principal should not use a single-prized tournament for the purpose of inducing the optimal levels of efforts from the agents. Single-prized tournament 'bundles' the tasks and outputs together, and such a tournament will not work in general. Instead, the principal should use task-specific multi-prized tournaments that are tailored for each task. The main reason that this tournament design works in these contexts is that the principal has extra degrees of freedom to adjust the sizes of the prizes needed for delegating the right incentives to the agents.

Our study is theoretical. As we have already noted in the Introduction, there are several occasions where the contexts similar to those modeled in this paper arise. It would be interesting to see how our model and theoretical results fare in such occasions.

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